**NUMERICAL METHODS
FOR MATHEMATICAL PHYSICS INVERSE PROBLEMS**

## Lecture 3. Differentiation of functionals

We know that the inverse problems can be transformed to the problems of extremum finding. So the practical methods of inverse problems theory are based on the optimization methods. We considered methods of analysis for the easiest extremum problem. It was the problem of the minimization for the function of one variable. We analyze it with using of the stationary conditions and gradient method. However the practical inverse problems can be transformed to the optimization problem for the functions of many variables and the functionals even. So we would like to extend the known minimization methods to these problems. Therefore our next aim is the determination of the differentiation for the general functionals.

### 3.1. Gateaux derivative

Let us consider a general functional *I* on the arbitrary set *V*. We could extend in principle the stationary condition and the gradient method for its minimization if we had some methods of differentiation for this functional. Let us try to use the standard technique for calculate its derivative at a point *v* of the set *V.* It is know that the derivative of a function at a point is the result of passing to the limit to the increment of the function devised by the increment of the argument as the increment of the argument tends to zero.

Try to consider the increment of the argument at the point *v*. Let it be described by a value *h.* So the corresponding increment of functional is the difference between its value at the new point  and its value at the initial point *v.* There is the question, what is the sum of two elements of the given set *V*? We have the necessity to guarantee belonging of the sum of two arbitrary elements of *V* to this set. So the operation of the addition is determined on the set *V*. The difference  has the sense in this case because the difference between two numbers (values of the functional) is a number.

Our next step is the calculation of the ratio of the increments  We have the functional *I*, so its numerator is a number. But what is the sense of the division of the number  to the element *h* of the set *V*. It is clear, if *h* is a number. However it can be a vector or a function. Unfortunately the considered fraction does not have any sense in this case. So we need to correct the definition of the derivative.

We can determine the derivative of the function  at the point *x* by the following method. We consider the ratio  where *σ* and *h* are number. Then we try to pass to the limit here as *σ* tends to zero. If there exists a limit of this value, and this limit is linear with respect to *h*, then it has the form  for all value *h*,where  is the usual derivative of the function *f* at the point *x*.

We can try to use this idea for our case. Consider the ratio  where *σ* is a number, and *h* is an element of the set *V*. We have the ratio of two numbers here. So this term has the natural sense, and we will not any problem with division. However we need to interpret the product . Our technique can be true if we can determine the product between an arbitrary number *σ* and an arbitrary element *h* of the set *V*. So the operation of the multiplication of the element of *V* to the number is determined on the set *V*.

**Definition 3.1**. *The set with addition of the elements and the multiplication of the element to the number* (with natural additional properties) *is called the* ***linear space***.

Hence we will consider the set *V* as a linear space. The term  has the concrete sense for this case. Our next step is passing to the limit at the last term. However the convergence does not determined on the arbitrary set. We can pass to the limit on the *topological space* only. So we suppose that our set *V* is a topological space. But we have the necessity of a relation between the given algebraic operations and passing to the limit. Particularly we would like to have the convergence  for all *h* as  Then we would like to obtain the convergence  We can guarantee these properties if our operations are *continuous*.

Let us have the convergence  The operation of the multiplication by number is continuous if we obtain  Let us have also the convergence  The operation of the addition is continuous if we get 

**Definition 3.2**. *If the set is linear and topological space with continuous operations, then it is called the* ***linear topological space***.

Hence we will consider the set *V* as a linear topological space. So the passing to the limit for the term  as  has the sense. Suppose the existence of the corresponding limit. It depends from parameters *v* and *h* of course. We suppose also that its dependence from *h* is linear. So we have the equality

  (3.1)

where the map  is linear. The term in the left side of this equality is a number, because it is the limit of the ration of the two numbers. So the term in the right side of (3.1) is number too. Therefore the map  is a linear functional.

 **Definition 3.3**. *A functional I is called* ***Gateaux differentiable*** *at the point v if there exists a linear functional*  *such that the equality* (3.1) *holds. Besides*  *is called* ***Gateaux derivative*** *of the functional I at the point v.*

### 3.2. Examples of Gateaux derivatives

Consider examples of Gateaux derivatives.

**Example 3.1**. *The function of one variable.* Consider a function  Suppose it is differentiable at the point *x.* So we have the equality

  (3.2)

for all number *h*. So Gateaux derivative of the function *f* at the point *x* is its classical derivative at this point.

**Example 3.2**. *The function of many variables.* Consider a function  Determine its Gateaux derivative at the point  We would like to pass to the limit in the equality



as  for all vector  Let the function *f* be differentiable with respect to all its arguments. Then we have

  (3.3)

The term at the right side of this equality is the scalar product between the *gradient*



and the vector *h*. So Gateaux derivative of the function *f* of many variable at the point *x* is its gradient at this point.

**Example 3.3**. *Linear functional.* The functional *I* on the set *V* is linear if it satisfies the following equality



for all elements *u* and *v* of *V* and all number *α,β*. Find the derivative of the linear functional *I* at the arbitrary point *v*. We have



for all *h*. So the linear functional is Gateaux differentiable at the arbitrary point; besides its derivative is equal to the initial functional.

**Example 3.4**. *Affine functional.* The functional *I* on the set *V* is called affine if it is determine by the equality



where *J* is a linear on *V*, and *a* is an element of *V*. We find



So the affine functional is Gateaux differentiable at the arbitrary point; besides its derivative is equal to the corresponding linear functional.

**Example 3.5**. *Lagrange functional.* Consider the functional



on the set *V* of the smooth enough functions on the interval  with zero values on the boundary of this interval. This functional is called Lagrange one. Let the function *F* be smooth enough. We find



where  as . Then we calculate



for all function *h* from the set *V*. Differentiating by parts, we get



because the function *h* is equal to zero at the point *x*1and *x*2. Therefore the derivative of Lagrange functional at the point (function) *v* is determined by the equality

  (3.4)

 **Example 3.6.** *Dirichlet integral*. LetΩbe *n-*dimensional set with the boundary *S.* Consider an integral



where *f* is a given function. This integral is called Dirichlet one*.* We consider it on the set *V* of the smooth enough functions on Ω with zero value on the boundary *S*. Find the value



Then we have



Using Green formula we get



where Δ is Laplace operator, and  is the derivative of *v* with respect to the exterior normal to the boundary *S*. So we can determine Gateaux derivative of Dirichlet integral is determine by the equality

  (3.5)

because the function *h* is equal to zero on the set *S*.

**Example 3.7.** *Discontinuous function*. Consider the function of two variables, which is determine by the equality



Find the difference



for all numbers *h* and *g*. After division by *σ* and passing to the limit as  we obtain that Gateaux derivative of the function *f* at zero point is zero, namely second order zero vector. However we continue our analysis. Determine  where *t* is a parameter. Then we have  Passing to the limit as  we have  and But the value of the function *f* at zero is zero. So this function is discontinuous at zero. Hence the discontinuous function of two variables can be Gateaux differentiable.

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**Example 3.8.** *Absolute* *value*. Consider the function  We have



Then we obtain



The result depends from the method of passing to the limit, and the result is not linear with respect to *h*. So the absolute value function is not Gateaux differentiable.

### 3.3. Gateaux derivatives for functional on unitary spaces

We determined that Gateaux derivative is definite by scalar product for the functions of many variables. It is possible to extend this result for the general linear spaces with scalar product.

**Definition 3.4**. *The value*  *is the* ***scalar product*** *of the elements u and v of the linear space V, if the maps*  *and*  *are linear functionals and*  *for all*  *besides*  *and*  *if and only if u is zero element of the space V, that is*  *for all v. The linear space with a scalar product is called the* ***unitary space****.*

The Euclid space  is the space with standard scalar product.

We can determine the convergence for the space with scalar product with following notions.

**Definition 3.5**. *Let V be the unitary space. The value*



*is called the* ***norm*** *of the element v of V.*

**Definition 3.6**. *The sequence*  *of the linear space V with scalar product converges to the point v, if*



The linear space with this convergence is the linear topological space.

**Definition 3.7**. *A functional I on the unitary space V is called* ***Gateaux differentiable*** *at the point v if there exists an element*  *of V such that*

  (3.6)

*holds. Besides*  *is called* ***Gateaux derivative*** *of the functional I at the point v.*

The set of real number R is the unitary space with scalar product



Then the formula (3.2) for Gateaux derivative of a function of one variable can be transformed to the equality (3.6).

Euclid space R*n* is the unitary space with scalar product



So the formula (3.2) for Gateaux derivative of a function of *n* variable can be transformed to the equality (3.6).

The set  of the continuous functions on the interval  with zero values at the points *x*1 and *x*2 is the unitary spaces with scalar product



The analogical scalar product can be determine for the space of the smooth enough functions with zero values at the boundary points of the given interval. So the equality (3.4) for the derivative of Lagrange functional can be transformed to the equality



Then we find Gateaux derivative



The set  of the continuous functions on the on the set Ω with zero values on the boundary of this set is the unitary spaces with scalar product



The analogical scalar product can be determine for the space of the smooth enough functions with zero values at the boundary points of the given set. So the equality (3.5) for the derivative of Dirichlet integral can be transformed to the equality



Then we find Gateaux derivative



**Example 3.9.** *Square of the norm*. Consider the functional



on the unitary space *V.* Find the value



Then we have



After division by *σ* and passing to the limit as ** we get



So 

### Task

Find Gateaux derivatives for the given functions of two variables and integral functionals:

|  |  |  |
| --- | --- | --- |
| variant | function  | functional |
| 1 |  |  |
| 2 |  |  |
| 3 |  |  |
| 4 |  |  |

### Next step

We know that inverse problems can be transformed to the minimization problems. The easiest minimization problem is the problem of the minimization for the function of one variable. It can be solve with using of the stationary condition and gradient method. This technique is based on the differentiation of the given function. Now we are able to differentiate general functionals. Then we try to apply the known optimization methods to problems of minimization general functionals.